THE REAL BEHAVIOUR OF COHESIONLESS GRANULAR MATERIALS UNDERGOING DEFORMATION. AN EXPERIMENTAL TECHNIQUE FOR MEASUREMENT OF STRESS-STRAIN DISTRIBUTION IN GRANULAR SOLIDS UNDER PRANDTL'S BOUNDARY CONDITIONS*

V. ŠMÍD and J. NOVOSAD

Institute of Chemical Process Fundamentals,
Czechoslovak Academy of Sciences, 165 02 Prague 6 - Suchdol

Received April 16th, 1975

An apparatus is described for the investigation of the real behaviour of granular solids undergoing deformation. The apparatus satisfies the boundary conditions of Prandtl's solution. The theoretical stress field is determined and the measuring technique is described. The obtained experimental stress distribution is compared with the theoretical solution and the deviations from the ideal behaviour of the granular solid are pointed out.

Measurement of stress and strain in mechanics of particulate solids has been developed primarily for the purpose of studying the motion of solids in vertical vessels. The methods chiefly used are the fixation of flow patterns and photographing of plane cuts[1-3]. This latter method yields essentially the pattern of particle trajectories. Photography has also been used to solve various problems in soil mechanics[4]. The most advantageous method of observing deformation appears to be the application of X-rays or γ-radiation[5]. A number of detectors have been developed to measure stresses at the walls and within granular solids. However, these latter devices interfere with the behaviour of the solid material in one way or another.

In order that the experimental results may provide a suitable basis for theoretical studies the following requirements have been put on the experimental technique: *a*) the ability of simultaneous measurement of the stress and strain of the granular material; *b*) the ability of achieving large deformations in order to reach and advanced stage of strain, *c*) simulation of the boundary conditions of the exact theoretical solution or a sufficiently close approximation to facilitate comparison of the real and theoretical behaviour and evaluation of the deviations.

Analysis of several cases has shown that the above requirements are best met by the set-up satisfying the boundary conditions of Prandtl's solution whose principle and theory has been presented in the preceding part of this series[6].

---

[*] Part IX in the series Studies on Granular Materials, Part VIII: This Journal *41*, 1740 (1976).

Experimental Set-Up

The function of the experimental set-up consists in constraining the material between two slabs of a wedge-shaped space, the boundaries of which satisfy the boundary conditions of Prandtl's solution. The design of the experimental apparatus is shown schematically in Fig. 1. Frame **1** supports two swivel seated slabs **2** and **3** making an angle α. The motion of these slabs is controlled by a screw **7** with a left-hand and right-hand thread. The front and the rear face of the wedge-shaped space is confined by glass walls **5**. The width of the slabs is taken to be approximately equal to the height of the heap of the granular material so as to satisfy the assumption of plane stress. In order that we may keep the orientation of the friction forces as required by Prandtl's solution[6] there is a thin plate **4** traversing over the surface of slab **2**. The motion of the plate **4** is derived from the motion of the principal slabs **2** and **3** by means of a guide **6**. The speed of motion of the plate **4** is such that throughout the whole closing operation it is greater than the speed of displacement of the granular material. The surface of the slab **3** and the plate **4** is covered by a layer of granular material cemented to it. This makes the wall friction, ϕ_w, identical with the angle of internal friction ϕ.

In the course of the deformation the stress of the granular material is measured at the wall by means of a pressure cell **9**. The surface of the sensing plate in the pressure cell is covered similarly as the surface of the wedge.

The velocity field is recorded photographically by a camera **10** mounted on a stativ **11** connected to the revolving slab **2**. Firm coupling of the camera and the slab **2**

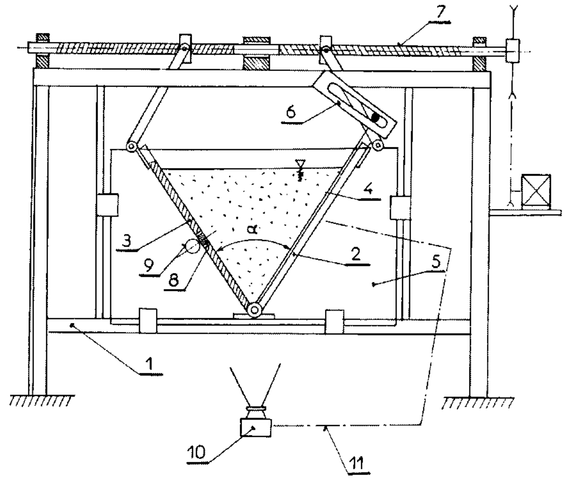

FIG. 1

Scheme of Experimental Set-Up

facilitates evaluation since the slab may then serve as a frame of reference for the coordinate system $0, r, \theta$. The photographs thus reflect the relative motion of the solid particles with respect to the slab. Longer exposure then gives rise to the streamlines.

The pressure cells have been developed to measure wall stresses in bunkers and have been described earlier[7,8]. The principle of the pressure cell is shown in Fig. 2. For a proper function of the cell it was essential that no granular material penetrate into the space between the sensing plate **8** and housing **12**. A very effective means for sealing this slot proved to be self-vulcanizing rubber **15** which did not affect adversely either the sensitivity or the elastic characteristics of the cell.

Experimental Method

The experimental wedge is filled in the open position with a granular material and the heap is levelled. Turning on the drive of the screw **7** the slabs **2** and **3** begin to move and simultaneously plate **4** is pulled out.

The motion of the deformed material is recorded photographically. The time of exposure had been selected in advance depending on the angle of the wedge. The stress is measured by the pressure cell throughout the deformation process and recorded by a line recorder.

In order to ensure reproducible initial conditions each sample of granular material was subjected before each experiment to at least two deformation cycles consisting of opening and closing the wedge, which removes eventual inhomogeneities of the

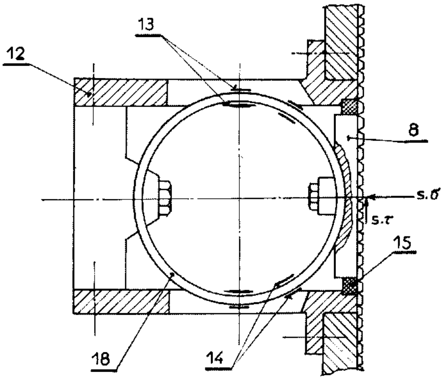

FIG. 2

Pressure Cell

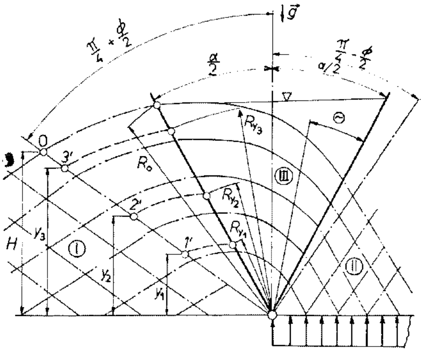

FIG. 3

Prandtl's Wedge on Transition between Passive I and Active II Region of Granular Material Oriented Identically with Experiment sample caused by filling and levelling. For the same reason the top surface of the sample material between individual deformation cycles is not levelled off. This also shows the extent to which the conditions of plane stress assumed in the theory are met. The fact that the shape of the top surface of the sample does not vary appreciably during deformation and, above all, the top is a ruled surface parallel to the axis passing through the apex of the wedge indicates that the conditions of plane deformation are well met and that the pattern of streamlines taken through the glass walls is identical with that within the material.

Theoretical Stress Distribution at the Measuring Point

To determine the theoretical stress in the examined position we make use of the fact[9,10] that for a uniform and continuous load on a part of the half space occupied by the granular material, Prandtl's solution forms the transition region III between the active II and passive stress region I as shown in Fig. 3.

For a point on the fictious limits between the passive and the transition region characterized by a height H and position of the pressure cell y, we may write for the mean stress

$$p' = \varrho_s g (H - y) \frac{1 + k}{2} . \qquad (1)$$

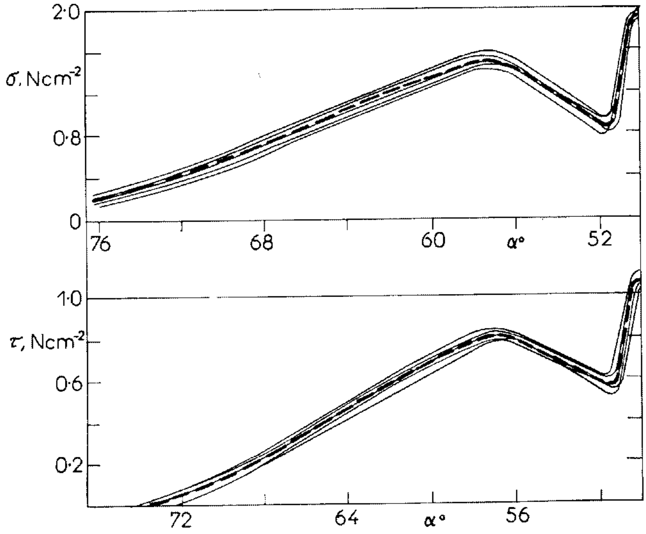

FIG. 4
Experimental Stress Behaviour

On the basis of Prandtl's solution of the sliding lines[6,9,11] we have for the fictious height H or an arbitrary y_i that

$$H = R_0 \exp\left[\left(\frac{\pi}{4} + \frac{\phi}{2} - \frac{\alpha}{2}\right) \operatorname{tg} \phi\right] \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right),$$

$$y_i = R_i \exp\left[\left(\frac{\pi}{4} + \frac{\phi}{2} - \frac{\alpha}{2}\right) \operatorname{tg} \phi\right] \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right). \qquad (2)$$

The same solutions yields for the change of the mean stress[6] along the sliding line corresponding to an angular displacement $\Delta\theta$ the relation

$$p_{r,\Theta} = p_{r_0,\Theta_0} \exp\left(2 \Delta\theta \operatorname{tg} \phi\right). \qquad (3)$$

Combining Eqs (*1*)–(*3*) a relation for the mean stress at the point of the pressure cell is obtained

$$p = \varrho g(R_0 - R_i) \exp\left[3\left(\frac{\pi}{4} + \frac{\phi}{2} - \frac{\alpha}{2}\right) \operatorname{tg} \phi\right] \frac{1 + k}{2} \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \qquad (4)$$

and then the normal and tangential components of the stress can be expressed as

$$\sigma = p \cos^2 \phi\,, \quad \tau = p \cos \phi \sin \phi\,. \qquad (5)$$

Both components of the stress may then be expressed as functions of the angle α made by the slabs of the Prandtl's wedge.

EXPERIMENTAL

The granular material used for experiments was river quartz sand marketed for water filtration. Particle diameter distribution was determined by a screen analysis and the Rosin–Rammler–Sperling equation was used for description of the distribution[12]. According to this formula the percentual residual on a sieve is

$$D = 100 \,.\, \exp\left[x/\bar{x}\right]^n. \qquad (6)$$

The mean statistical diameter of the grain was $\bar{x} = 1{\cdot}7$ mm and the exponent of polydispersity was $n = 5{\cdot}54$.

The angle of internal friction was determined on the Jenike shear cell[13]. The measurements involved on one hand uncompacted material compressed only by the static force ($n_{tw} = 0$) and, on the other hand, the material compacted by twisting consolidating lid ($n_{tw} = 40$). The value of the angle of internal friction found for the uncompacted sample was $\phi_{min} = 28°$, while the same sample after compacting displayed the value $\phi_{max} = 32°30'$. In terms of the mean value the angle of internal friction is $\phi = 30°15' \pm 2°15'$ which is equal to the effective angle of internal friction, δ, as we deal with a cohesionless material. The bulk density was also determined on the Jenike cell with the result $\varrho_s = 1{\cdot}64\ \text{g/cm}^3$ for the compacted sample. The freely dumped sample exhibited the value $\varrho_{s\,\min} = 1{\cdot}55\ \text{g/cm}^3$. The mean bulk density is thus $\varrho_s = 1{\cdot}602\ \text{g/cm}^3$.

Experimental values. The deformation angles of the wedge ranged between 76°30′ and 50°. The radius vector of the pressure cell was $R = 330$ mm measured from the apex of the wedge. The cell recorded the course of the normal and the shear component of the stress. The characteristic dimension of the apparatus is given by the radius R_0 of the height of the sample, which for the maximum angle of opening equalled 545 mm. The width of the wedge in the direction of the axis of revolution was $l = 504$ mm. Reproducibility of the initial conditions was ensured by two deformation cycles consisting of opening and closing the wedge performed before the actual experimental run. A chart of the recorded normal and shear components of the stress in the course of closing the wedge is shown in Fig. 4. The arithmetic mean of all measurements was taken as the typical course of both stress components (in Fig. 4 shown by broken line) and used for comparison with the theoretical solutions.

The experimental results were compared with the theoretical values calculated from Eqs (*4*) and (5).

Fig. 5 shows in its upper part the course of both components of the stress as a ratio of the experimental/theoretical value as functions of the relative opening of the wedge.

The bottom part of Fig. 5 is a plot of the true coefficient of internal friction of the material given by the ratio of the tangential and the normal component of the stress on the sliding surface in the course of deformation.

The coefficient of friction is also shown as a ratio related to the friction coefficient determined by the shear cell tests.

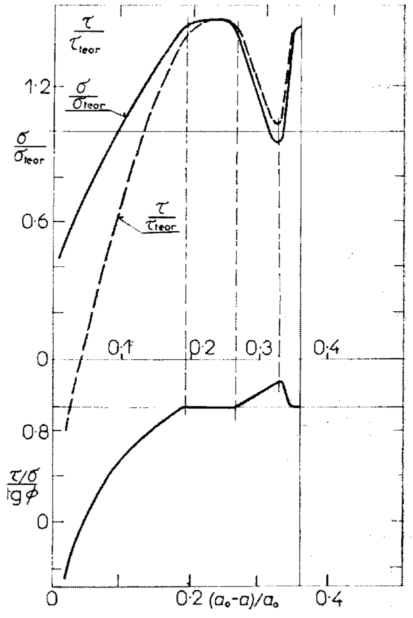

FIG. 5

Comparison of Experimental and Theoretical Stresses

DISCUSSION

The described experimental technique satisfies the requirements listed in the introductory part. At individual time instants the velocity field is known and the pictures can be shot sufficiently rapidly in sequence in order to obtain the image of the true velocity field throughout the process of deformation in the Prandtl's wedge. The stress can be continuously recorded and measured in an arbitrary position on the slab forming the wedge or eventually in several positions at the same time. Considerable degree of deformation can be achieved and for the given type of deformation an approximate stress and velocity field can be obtained theoretically.

The apparatus has been designed for relatively small rates of deformations from the viewpoint of industrial operations. However, it may be assumed that greater rates of deformations would not cause deviations in the behaviour of the material.

The obtained theoretical distribution of stress and deformation and the true value of these quantities can be used to assess the validity of various stress-strain relations and to determine eventual deviations caused by the real behaviour and real properties of granular solids.

A comparison of experimental results with the theoretical solution of the stress field has shown numerous problems of the mechanics of particulate solids. First of all it is the relatively large deformation which is required to reach the limiting state of stress corresponding to the given deformation in dependence on the angle of opening of the wedge. To achieve this limiting stress required a 19% change from the original angle of the Prandtl's wedge (a similar deformation has been found necessary by Roscoe and Burland[14] who found transition from the active into the passive state to be associated with a 18% change in the direction of the principal stress). The large deformations necessary to attain the expected limiting state suggest that there is a possibility of a gradual mobilization of limiting stresses in dependence on the magnitude of the deformation path. This offers the possibility of reaching the limiting state gradually from positions with larger deformations toward those with a shorter deformations. However, the limiting state of stress corresponding to the given way of deformation was not achieved until the experimental stresses exceeded the theoretical ones by about 45%.

In the next stage of deformation (19–26%) the coefficient of internal friction assumed a constant value corresponding to the lower value of the angle of internal friction determined by the shear cell tests. Also the relative values of the stress were approximately constant and equal to 1·45 times the theoretical value. In the subsequent stage of deformation (26–35%) a very surprising phenomenon was observed. Instead of the expected monotonous behaviour a marked reproducible pulse of stress appeared. The amplitude of the pulse was bounded from below by the theoretical value and from above by a value exceeding the theoretical one by 45%. The stress at the end of the pulse with respect to the theoretical value was thus in the same ratio as in the preceding stage of deformation. It is further worth noting that the decrease of the stress was approximately three times slower than the subsequent increase.

The observed stress changes were accompanied by changes of the coefficient of internal friction of the material. The friction coefficient grew from the initial value up to that corresponding to the upper limit of the angle of internal friction, determined by the shear cell tests for the compacted sample, and subsequently fell back to the original value.

Judging from this fact it seems that even in the advanced stage of deformation a certain strengthening of the material may occur as a consequence of the plastic deformation. Such strenghtening would then show as a change of the slope of the straight line envelope of the Mohr's circles or eventually by its curvature.

Shortly after the pulse the apparatus reached the upper limit of its deformation range which amounted to 36·5% of the initial opening of the wedge. It was therefore not possible to observe the behaviour of the granular material further and verify possible fluctuation of the stress and of the coefficient of internal friction of the material.

Nevertheless, the experiments show that controlled plastic deformation gives rise both in the initial and the advanced stage of deformation to various interesting phenomena which require additional more detailed study. The problem particularly important for description of the real behaviour of granular materials seems that of mobilization of the limiting state of the plastic stress as well as the problem of possible fluctuation of the stress and associated strengthening of the material in the advanced stage of deformation.

LIST OF SYMBOLS

D	residue on the screen
g	acceleration due to gravity
H	fictious height of sample in region of passive stress
k_t	ratio of principal stresses in granular material
n_w	number of twists of consolidating lid
p	mean stress in granular material
r	radial coordinate
R_i	radius of point of observation
R_0	radius vector of surface
R_y	radius vector of position of pressure cell
y_i	fictious height of point of observation in region of passive stress
ϕ	angle of internal friction
ϱ_s	bulk density of material
σ	normal stress
τ	shear stress
θ	radial coordinate

REFERENCES

1. Dubynin N. N. G.: *Kinematics and Dynamics of Bulk Solids During Discharge from Orifices.* ASME Publ. No 68-MH-27, Boston Symp. Flow Solids, Boston 1968.
2. Kvapil R.: *Teorie toku sypkých a balvanovitých hmot v zásobnících.* Published by SNTL, Prague 1955.
3. Novosad J., Surapati K.: Powder Technol. *2*, 82 (1968/69).
4. de Josselin de Jong G.: Proc. Conf. Earth Press. Probl., p. 55. Brussel 1958.
5. Bransby P. L.: Presented at Symp. EFCE, London, September 1973.
6. Šmíd V., Novosad J.: This Journal *41*, 1740 (1976).
7. Šmíd J., Novosad J.: Powder Technol. *4*, 322 (1970/71).
8. Šmíd J., Novosad J.: This Journal *37*, 3568 (1972).
9. Golushkevich S. S.: *Statika Predelnych Sostoyanii Gruntovych Mass.* Gos. Izd. Tekh.-Teor. Lit., Moscow 1957.
10. Sokolovskii V. V.: *Statika Sypuchei Sredy.* Gos. Izd. Fyz.-Mat. Lit., Moscow 1960.
11. Prandtl L.: *O Tvardosti Plasticheskikh Materialov i Soprotivlenii Reazaniu.* Sb. Teoria Plastichnosti IL. Moscow, 1948 (Russian translations).
12. Anděl A.: *Základní pochody chemické a potravinářské výroby. Pochody mechanické I.* Vydavatelství ČVUT, Prague 1969.
13. Jenike A. W.: *Storage and Flow of Solids.* Bull. Univ. Utah, No 123, Vol. 23 (1969).
14. Roscoe K. H., Burland J. B. in the book: *Engineering Plasticity* (J. Heyman, F. A. Leckie, Eds). Cambridge Univ. Press 1968.

Translated by V. Staněk.

V. Šmíd and J. Novosad:

The Real Behaviour of Cohesionless Granular Materials Undergoing Deformation

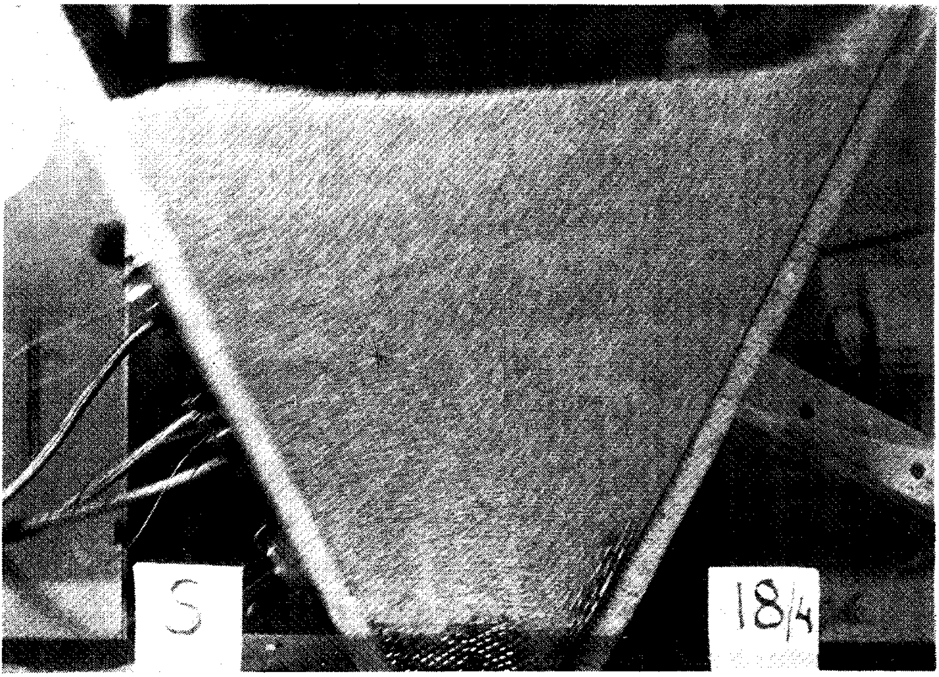

Fig. 2

Photograph of Streamlines